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Nuclear Calabi–Yau Space

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In this note a Calabi–Yau manifold already found for ^{9}Li will be shown to carry an Euler number of six if Yang–Mills symmetry is broken. Not only does this specify the correct number of generations of quarks and leptons, but peaks on the manifold are associated with the lowest eigenvalues of a CP-invariant Dirac spin operator $C_{[\Lambda]}$.

1. INTRODUCTION

If one seeks a configuration of the form $M^4 \times K$ where M^4 is four-dimensional Minkowski space and K a compact Riemannian six-dimensional manifold, then the only way of modeling the space-time geometry of superstrings is for K to be a Calabi–Yau space, specifically, a compact three-dimensional complex manifold with a Ricci flat Kähler metric.

De Wet (1995) has shown that a spin manifold that carries an odd number of fermions is Calabi–Yau with positive sectional curvature, and ^{9}Li was taken as an example where it was found possible to calculate the metric tensor and therefore the connections which are the Yang–Mills gauge field responsible for the strong, weak, and electromagnetic interactions. This manifold will also be found to be a twistor space according to the criteria of Lawson and Michelsohn (1989, Chapter IV, §9). The twistor can be shown to be a vibrating torus by making use of a theorem introduced in Section 2 that enables one to exponentiate a CP-invariant Dirac operator $C_{[\Lambda]}$ that is a function of the spins and parities of any odd-A nucleus.

Closed geodesics can be generated from the harmonics of $\exp(C_{[\Lambda]}\theta)$ as shown in Fig. 1 which depicts a quadrapole.

Normally the Dirac matrix is a differential operator S such that the Laplace–Beltrami operator Δ is S^2 and the zero modes of S are the harmonic

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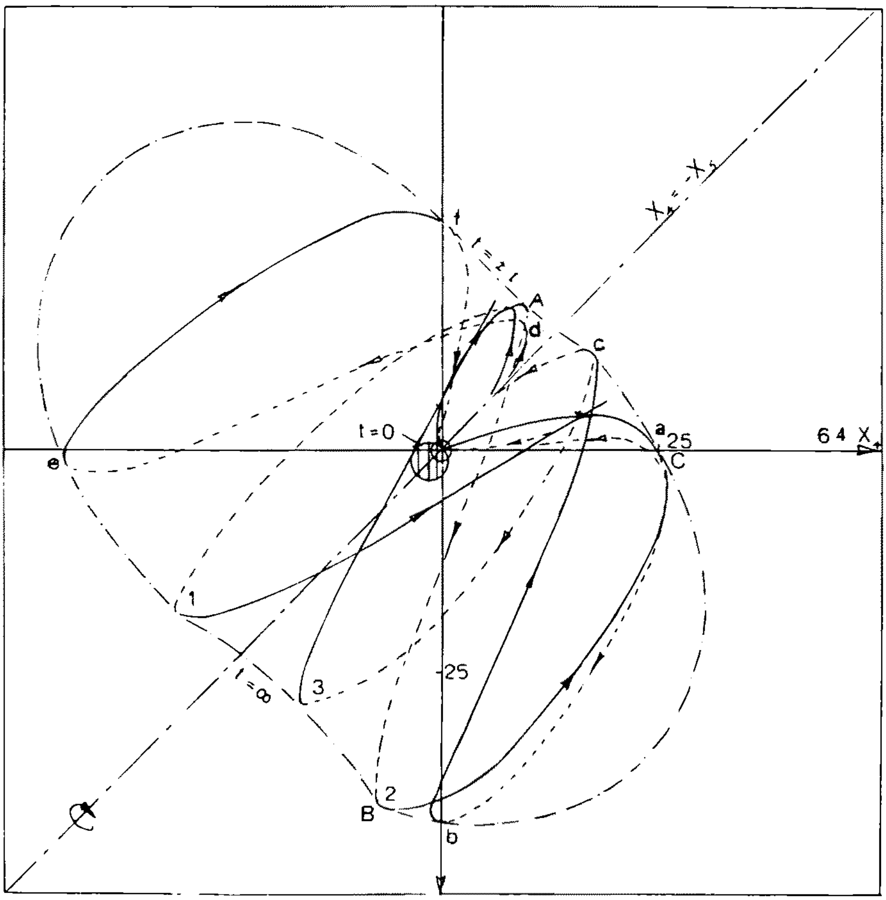

Fig. 1. Geodesics on the manifold of ^{9}Li, ^{9}C.

differential forms ω^p given by $\Delta\omega^p = 0$. The number of such p forms is the Betti number b_p, from which it follows that the index of S is half of the Euler characteristic χ of the manifold (see, for example, Green *et al.*, 1988, Chapter 14).

Although the CP-invariant operator $C_{[\Lambda]}$ is not a differential operator, the $\exp(C_{[\Lambda]}\theta)$ are harmonics, so it makes sense to determine the Euler characteristic of the associated spin manifold and see how χ is related to the lowest eigenvalues of $C_{[\Lambda]}$. This is the purpose of this paper, because $\chi/2$ is also the number N_{gen} of fermion generations identical in their gauge quantum numbers and should be three.

Actually in the case of a torus $\chi = 0$, but by examining the curvature, peaks or "horns" can be found associated with the three lowest spins of $C_{[\Lambda]}$

(which label the lowest energy levels). These could represent instantons that become quarks or leptons at energies sufficiently high to break Yang–Mills symmetry. Then the horns would be singularities with infinite curvature and the manifold would become an orbifold with an Euler characteristic of three. One possible point is numbered 2 in Fig. 2 and a closed loop around this singularity would be trapped and add an Euler characteristic of unity. The point 2 can also rotate about $X_4 = -X_5$ (Fig. 1) to yield another Euler number and there are two similar points 1 and 3, positioned in Fig. 1, to make a total of six. Physically the curvature is a measure of the strength of the Yang–Mills field, which becomes infinite when the symmetry is broken, and quarks and leptons appear at sufficiently high energies. The sectional curvature of the ^{9}Li manifold is calculated in Section 3.

2. THE MATRIX EXPONENTIAL THEOREM

It was shown by de Wet (1994) that the tensor product of self-representations of the quaternion Dirac ring decomposes into left modules $C_{[\lambda]}P_{[\lambda]}$ each of which represents a nuclear state $[\lambda]$ with a definite spin, charge, and parity. This is called a Dirac bundle by Lawson and Michelsohn and is of course an example of the nuclear spectral theorem. Specifically, $[\lambda]$ is a partition

$$A = \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4, \qquad [\lambda] \equiv [\lambda_1\lambda_2\lambda_3\lambda_4]$$

of the atomic number such that $(\lambda_3 + \lambda_4)$ is the number of nucleons with positive charge, $(\lambda_2 + \lambda_3)$ the number with a given spin σ, and $(\lambda_2 + \lambda_4)$ the number with a given parity π. The possible states of ^{9}Li, ^{9}C, which may be shown to be coherent, are set out in Table I and these label the rows of the (1, 1) form (2.3).

The operators $C_{[\lambda]}$ may be expressed in terms of the generators

$$\sigma_i = E_N \otimes {}^P\Gamma_i + {}^N\Gamma_i \otimes E_P,\ \pi_i = E_N \otimes {}^P\Gamma_i - {}^N\Gamma_i \otimes E_P, \qquad i = 1, 2, 3 \tag{2.1}$$

of O_4. Here ${}^P\Gamma_i$, ${}^N\Gamma_i$ are $(P + 1)$-, $(N + 1)$-dimensional Lie operators of SO_3; E_P, E_N are $(P + 1)$, $(N + 1)$ unit matrices, and σ_i is the spin angular momentum matrix for a coupled system of P protons and N neutrons. It is also an example of the natural decomposition of a Clifford algebra (Lawson and Michelsohn) and is why the nucleus can be modeled by quaternions.

It is then possible to find the specific operator $C_{[\Lambda]}$ which is also CP-invariant. If $\sigma_0 \equiv 2\sigma_1$ and $\pi_0 \equiv 2\pi_1$, then an example is the Wigner series

$$^9\text{Li:} \quad C_{[3303]} = \frac{1}{6}(\sigma_0^3 + \pi_0^3) + \frac{3}{2}(\sigma_0\pi_0^2 + \sigma_0^2\pi_0) + \frac{17}{3}(\sigma_0 + \pi_0) \tag{2.2a}$$

$$^9\text{C:} \quad C_{[3033]} = \frac{1}{6}(\sigma_0^3 - \pi_0^3) + \frac{3}{2}(\sigma_0\pi_0^2 - \sigma_0^2\pi_0) + \frac{17}{3}(\sigma_0 - \pi_0) \tag{2.2b}$$

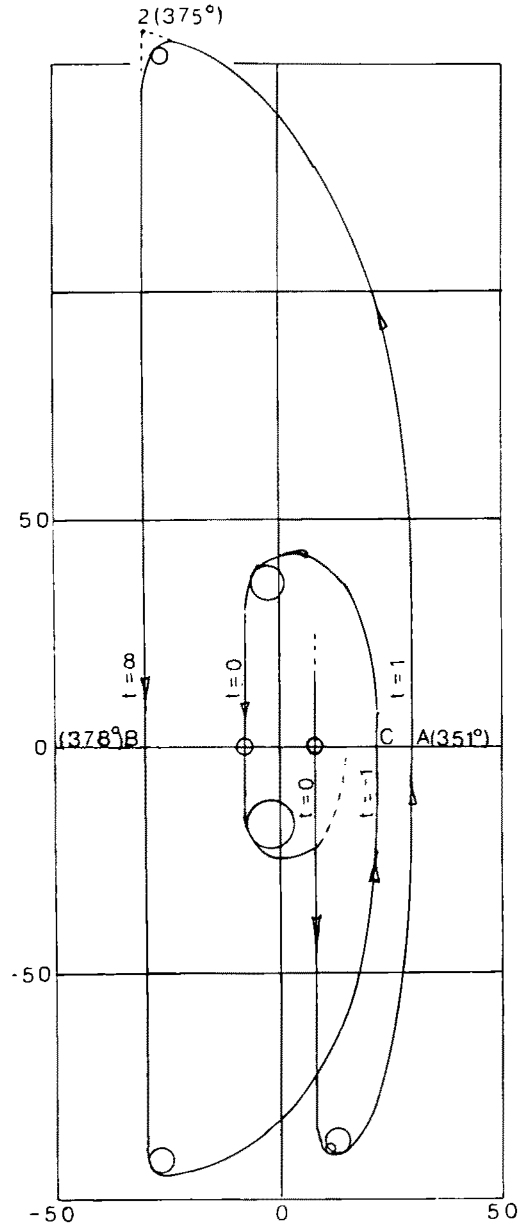

Fig. 2. Cross section under the geodesic OABCO.

Table I. Coherent States of ^{9}Li, ^{9}C

^{9}Li		^{9}C		^{9}Li		^{9}C			
$p = -$	$+$	$p = -$	$+$						
$s = +$	$-$	$s = -$	$+$						
$\lambda_1\lambda_2\lambda_3\lambda_4$	$\lambda_2\lambda_1\lambda_4\lambda_3$	$\lambda_3\lambda_4\lambda_1\lambda_2$	$\lambda_4\lambda_3\lambda_2\lambda_1$	σ_0	π_0	σ_0	π_0	$C_{[3303]} = C_{[3033]}$	$C_{[3033]}/16$
6003*	0630	0360	3006	$9i$	$-3i$	$9i$	$3i$	$160i$	$10i$
6012	0621	1260	2106	$7i$	$-5i$	$7i$	$5i$	$80i$	$5i$
6021*	0612	2160	1206	$5i$	$-7i$	$5i$	$7i$	$-80i$	$-5i$
6030	0603	3060	0306	$3i$	$-9i$	$3i$	$9i$	$-160i$	$-10i$
5103*	1530	0351	3015	$7i$	$-i$	$7i$	i	$40i$	$2.5i$
5112	1521	1251	2115	$5i$	$-3i$	$5i$	$3i$	$40i$	$2.5i$
5121*	1512	2151	1215	$3i$	$-5i$	$3i$	$5i$	$-40i$	$-2.5i$
5130	1503	3051	0315	i	$-7i$	i	$7i$	$-40i$	$-2.5i$
4203*	2430	0342	3024	$5i$	i	$5i$	$-i$	$-32i$	$-2i$
4212	2421	1242	2124	$3i$	$-i$	$3i$	i	$16i$	i
4221*	2412	2142	1224	i	$-3i$	i	$3i$	$-16i$	$-i$
4230	2403	3042	0324	$-i$	$-5i$	$-i$	$5i$	$32i$	$2i$
Λ 3303*	3330	0333	3033 Λ						
				$3i$	$3i$	$3i$	$-3i$	$-56i$	$-3.5i$
3321*	3312	2133	1233	$-i$	$-i$	$-i$	$+i$	$-8i$	$-\frac{1}{2}i$

which is manifestly CP-invariant because $T_3 \to -T_3$ is accompanied by $\pi_0 \to -\pi_0$. Moreover, the matrix representations are identical up to a rearrangement of rows and columns. If $(\lambda_2 + \lambda_3)$ is chosen to be the number of nucleons with a negative spin and $(\lambda_2 + \lambda_4)$ the number with positive parity, then

$$\sigma_0 = i(A - 2(\lambda_2 + \lambda_3)), \qquad \pi_0 = -i(A - 2(\lambda_2 + \lambda_4))$$

and the eigenvalues λ of (2.2), shown in the last three columns of Table I, may be evaluated directly. These label the rows of $C_{[\Lambda]}$ found by using the matrix representation (2.1) and because the eigenvalues are the same the state labeling is confirmed even though they are reordered.

In fact the matrix $C_{[\Lambda]}$ is generally reducible, but an irreducible submatrix μ may be determined that includes the eigenvalue Λ and if A is odd, μ is a (1, 1) form with the complex structure

$$\begin{bmatrix} 0 & B \\ -B & 0 \end{bmatrix} \tag{2.3}$$

where B is a real, symmetric $p \times p$ matrix with coordinates $k = \lambda_k\theta$. It is therefore the horizontal subspace of a complex Grassmann or Kähler manifold [Kobayashi and Nomizu, 1969, Chapter IX, example (6.4)], but before we can find the connections it is necessary to exponentiate using a theorem developed by de Wet (1994, 1995) which may be summarized as follows.

Let the eigenvalues of B be $\{\lambda_0; \lambda_1'; \lambda_2'; \ldots ; \lambda_n'\}$; then it is always possible to express them in the canonical form

$$\{0; 1; \lambda_2; \ldots ; \lambda_n\} \tag{2.4}$$

where $\{\lambda_2; \ldots ; \lambda_n\}$ are positive, by adding or subtracting an angular momentum λ_0 and then dividing by $\lambda_f = (\lambda_1 \pm \lambda_0)$. This follows because if $BX = \lambda X$, then

$$(B - \lambda_0)X = (\lambda - \lambda_0)X$$

It will appear below that exponentiation of the translated or canonical spectrum leads to a factor $e^{i\lambda_0\theta}$ that is responsible for vibrational modes and curvature, while λ_f may be absorbed in θ and does not change the shape of the geodesics although there is a frequency change. We now let (2.4) be the eigenvalues of μ and define the orthogonal functions

$$F(\mu) \equiv \mu(\mu^2 + 1)(\mu^2 + \lambda_2^2) \ldots (\mu^2 + \lambda_n^2) = 0$$

$$F_0(\mu) \equiv F(\mu)/\mu, \qquad F_k(\mu) \equiv F(\mu)/(\mu^2 + \lambda_k^2), \qquad F_j(\mu)F_k(\mu) = 0$$

Then

$$e^{\mu\theta} = \mu \sum_{k=0,1,\ldots}^{n} \frac{F_k(\mu) \cos \lambda_k\theta}{i\lambda_k F_k(i\lambda_k)} + i \sum_{k=1,2,\ldots}^{n} \frac{F_k(\mu)}{F_k(i\lambda_k)} \sin \lambda_k\theta \tag{2.5}$$

Proof:

$$\mu = \left.\frac{de^{\mu\theta}}{d\theta}\right|_{\theta=0} = \mu \sum_{k=1,2,\ldots}^{n} \frac{i\lambda_k F_k(\mu)}{F_k(i\lambda_k)\mu} = \mu \sum_{k=1,2,\ldots}^{n} K_k(\mu) \tag{2.6}$$

where

$$K_k(\mu) = \frac{i\lambda_k F_k(\mu)}{F_k(i\lambda_k)\mu} \tag{2.7}$$

is idempotent and $K_j(\mu)K_k(\mu) = 0$ (de Wet, 1995); therefore $\Sigma_k K_k(\mu)$ is a decomposition of unity and (2.6) follows. The $K_k(\mu)$ are projection operators onto the orthogonal states of $C_{[\Lambda]}$. The proof of (2.7) covers the case $\theta = 0$ and in the special case $\mu = i$, only

$$F_1(\mu) = i(i^2 + \lambda_2^2) \ldots (i^2 + \lambda_n^2) = F_1(i)$$

survives, since $\lambda_k = 1$, and we find the elementary circular relation

$$e^{i\theta} = \cos \theta + i \sin \theta$$

3. THE CURVATURE TENSOR

We may write (2.5)

$$e^{\mu\theta} = Z_0(\cos\theta) + Z_1(\sin\theta) = \begin{bmatrix} Z_0 & Z_1 \\ -Z_1 & Z_0 \end{bmatrix} \tag{3.1}$$

because Z_0 depends only on the even powers of μ and Z_1 only on the odd. Then, making use of the orthogonality condition $F_j(\mu)F_k(\mu) = 0$, it is easy to determine Z and hence find that

$$T \equiv Z_1 Z_0^{-1} = -T' = \mu \sum_{k=1,2,\ldots}^{n} \frac{i(F_k(\mu)/\mu)}{F_k(i\lambda_k)} \tan \lambda_k\theta \tag{3.2a}$$

$$T\overline{T}' = \sum_{k=1,2,\ldots}^{n} K_k(\mu) \tan^2\lambda_k\theta \tag{3.2b}$$

Now the metric on a complex Grassmann manifold is

$$ds^2 = \operatorname{Tr} \frac{dT}{(1 + T\overline{T}')} \frac{d\overline{T}'}{(1 + T\overline{T}')} \tag{3.3}$$

where $\overline{T}'$, $d\overline{T}'$ are the conjugate transposes of T, dT (Kobayashi and Nomizu, 1969, Chapter IX, §6; Wong, 1967). Using (3.2), this is simply the flat measure carried by a torus, namely

$$ds^2 = \sum_{k=1,2,\ldots}^{p} dz_k\, d\bar{z}_k, \qquad z_k = i\lambda_k\theta \tag{3.4}$$

However, a translation to the canonical form (2.4) introduces the factor tan $\lambda_0 t$, so (3.2) becomes

$$T = \mu \tan \lambda_0\theta \sum_{k=1,2,\ldots}^{n} \frac{i(F_k(\mu)/\mu)}{F_k(i\lambda_k)} \tan \lambda_k\theta \tag{3.5a}$$

$$T\overline{T}' = \tan^2\lambda_0\theta \sum_k K_k(\mu) \tan^2\lambda_k\theta \tag{3.5b}$$

(de Wet, 1995). The manifold is now conformal and we can find the regions of very large positive curvature. To do this, write (3.3)

$$ds^2 = g_{k\bar{k}}\, d(\lambda_k\theta)\, d(-\lambda_k\theta)$$

$$= d(\lambda_k\theta)\, d(-\lambda_k\theta) \sum_k \frac{\mu}{\lambda_k} K_k(\mu) g(\lambda_k\theta) \sum_k \frac{\overline{\mu}'}{\lambda_k} K_{\bar{k}}(\overline{\mu}') g(-\lambda_k\theta)$$

with

$$g(\lambda_k\theta) = -g(-\lambda_k\theta) = \tan \lambda_0\theta \sec^2\lambda_k\theta/(1 + \tan^2\lambda_0\theta \tan^2\lambda_k\theta) \tag{3.6}$$

Here $\mu = -\overline{\mu}'$, $K_{\bar{k}}(\overline{\mu}') = K_k(\mu)$, and $k = \lambda_k\theta$ are the coordinates of the matrix B in (2.3), $\bar{k} = -\lambda_k\theta$, and $\pm i\lambda_k\theta$ are the coordinates of μ. The only nonvanishing components of the affine connection are

$$\Gamma_b{}^k{}_k = g^{k\bar{k}} \frac{\partial}{\partial b} g_{k\bar{k}}, \qquad \Gamma_{\bar{b}}{}^{\bar{k}}{}_{\bar{k}} = g^{k\bar{k}} \frac{\partial}{\partial \bar{b}} g_{\bar{k}k} \tag{3.7}$$

with $b = \lambda_0\theta$, $\bar{b} = -\lambda_0\theta$ (Green *et al.*, 1988, §15.3.3). The manifold is Ricci flat and is a Calabi–Yau space, but the sectional curvature is

$$K = R_{k\bar{k}k\bar{k}} = \frac{\partial^2 g_{k\bar{k}}}{\partial k\, \partial \bar{k}} - \sum_p \frac{\partial^2 g_{e\bar{e}}}{\partial e\, \partial \bar{e}} \tag{3.8}$$

Then by (3.6)

$$\frac{\partial^2 g_{k\bar{k}}}{\partial k\, \partial \bar{k}} = \sum_k \left(\frac{\mu}{\lambda_k} K_k(\mu)\right)\left(\frac{\overline{\mu}'}{\lambda_k} K_k(\mu)\right) G^2(\lambda_k\theta) \tag{3.9}$$

where

$$G(\lambda_k\theta) = \frac{\partial g(\lambda_k\theta)}{\partial(\lambda_k\theta)} = \frac{\partial g(-\lambda_k\theta)}{\partial(-\lambda_k\theta)}$$

$$= \frac{2 \sec^2\lambda_k\theta \tan\lambda_k\theta \tan\lambda_0\theta\, (1 - \tan^2\lambda_0\theta)}{(1 + \tan^2\lambda_k\theta \tan^2\lambda_0\theta)^2}$$

The boundaries of Fig. 1 are characterized by $t \equiv \tan \lambda_0\theta = 0, \pm 1, \infty$, where $G(\lambda_k\theta) = 0$ indicates an infinite radius of curvature; otherwise in the case of ^{9}Li a direct evaluation of

$$\sum_p \left(\frac{\mu}{\lambda_k} K_k(\mu)\right)\left(\frac{\overline{\mu}'}{\lambda_k} K_k(\mu)\right)$$

using the matrix representation of de Wet (1995) yields unity for each of the k terms summed over the p diagonal elements $(\mu/\lambda_k)K_k(\mu)$. Therefore the second term of (3.8) is simply $-\Sigma_k\, G^2(\lambda_k\theta)$, but a small correction is introduced by the first term, where $(\mu/\lambda_k)K_k(\mu)$ are the coefficients a_k of the element 4,4 of (3.1), namely

$$\psi_{4,4} = \frac{1}{128}\left\{15 + \sum_{k=1,2,\ldots}^{7} a_k \cos \lambda_k\theta\right\} \cos \lambda_0\theta$$

$$= \frac{1}{128}\left\{15 + 18 \cos\frac{\theta}{2} + 10 \cos\frac{5}{6}\theta + 3 \cos\theta + \cos\frac{4}{3}\theta\right.$$

$$\left. + 6 \cos\frac{3}{2}\theta + 30 \cos\frac{5}{2}\theta + 45 \cos 5\theta\right\} \cos\frac{5}{3}\theta \tag{3.10a}$$

This follows from a straightforward application of (2.5) to ^{9}Li with the canonical eigenvalues {0; 1/2; 5/6; 1; 4/3; 3/2; 5/2, 5}. The correction is then just $-(a_k G(\lambda_k \theta)/128)^2$ for each k.

As stated in the introduction, there are six peaks, which occur at the points 1 = 540° ± 273°, 2 = 540° ± 165°, 3 = 540° ± 57° of Fig. 1 owing to rotational symmetry about $X_4 = -X_5$. If we assume that at these points the principal radii of curvature are both equal to R, then, to the scale of Fig. 1, $R = 64/K^{1/2}$. These radii are drawn to half scale near the peaks of Fig. 2, which may be considered to be a sketch of the cross section under a geodesic *OABCO*. Moreover, the contributions to K_2, K_1, and K_3 come almost exclusively from a single term $G(\lambda_k \theta)$, where λ_k is 1/2, 4/3, and 3/2, respectively. But reference to Table I shows that these eigenvalues label the ground state [Λ] and the first excited states of ^{9}Li with spins $\sigma_0/2i$ of 3/2, |1/2|. The smaller peaks at the origin O have contributions from several λ_k and possibly do not degenerate into points when Yang–Mills symmetry is broken. However, reference to (3.9) and Fig. 2 confirms that the point 2 is also characterized by $\lambda_k = 0$, being the intersection of a zero and a positive curvature, which is precisely the index requirement corresponding to broken symmetry.

Finally the closed geodesic of Fig. 1 is constructed by means of Corollary 2.5 of Kobayashi and Nomizu (1969, Chapter X), namely by exponentiation of (2.3) and then plotting the wave function

$$X_4 \equiv \psi_{4,12} = \left(\sin \frac{5}{3}\, \theta/128\right) \sum_{k=1,2,\ldots}^{7} a_k \sin \lambda_k \theta \qquad (3.10b)$$

against its conjugate

$$X_5 \equiv \psi_{5,12} = -X_4(1080° - \theta)$$

characterized by sign changes of a_3, a_4, and a_7. This geodesic lies over the peaks 0, 1, 2, and 3.

4. CONCLUSIONS

The spin manifold analyzed in this contribution is Kähler and Ricci flat. It is therefore a twistor space as defined by Lawson and Michelsohn (1989, Chapter IV, §9) and as such has a decomposition into vertical and horizontal subspaces on a tangent plane. This decomposition is (3.1), and (2.3) belongs to a horizontal subspace.

REFERENCES

De Wet, J. A. (1994). *International Journal of Theoretical Physics*, **33**, 1887.

De Wet, J. A. (1995). *International Journal of Theoretical Physics*, **34**, 1065.

Green, M. B., Schwarz, J. H., and Witten, E. (1988). *Superstring Theory*, Cambridge University Press, Cambridge.
Kobayashi. S., and Nomizu, K. (1969). *Foundations of Differential Geometry*, Wiley-Interscience, New York.
Lawson, H. B., and Michelsohn, M.-L. (1989). *Spin Geometry*, Princeton University Press, Princeton, New Jersey.
Wong Yung-Chow (1967). *Proceedings of the National Academy of Sciences of the USA*, **57**, 589.